$$\eta_o^B = 13°34'$$

$$\eta_e^B = 30°00'. \qquad (24)$$

The upper and lower bounds on the phase shifts obtained by the quadratic Kato method [10] with the exact $\vec{e}$ are

$$14°30' \leqslant \eta_o \leqslant 15°30'$$

$$31°8' \leqslant \eta_e \leqslant 32°18'.$$

IV. Conclusion

A formulation has been presented for determining lower bounds on the phase shifts of dielectric obstacles in irregular waveguides. Relatively close lower bounds have been obtained with a simple trial function. This is to be expected for this type of obstacle for the permittivity value considered because the higher order evanescent modes contribute little excess phase shift. Of course, the bound may be improved by going to a better trial function with variational parameters. The extension to multimode waveguides is immediate.

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Propagation of Magnetostatic Waves Along Curved Ferrite Surfaces

NEERAJ C. SRIVASTAVA

***Abstract*—Electromagnetic equations have been appropriately transformed and solved in order to investigate the propagation of magnetostatic waves in curved geometries. The results have been utilized to study magnetostatic propagation along the surface of a cylindrically curved slab of ferrite in the azimuthal direction. In the case of an unbacked- or a metal-backed slab, it is found that the effect of curvature is to slightly reduce the phase as well as the group velocity by a constant factor throughout the frequency range of allowed modes. However, under favorable conditions, the presence of a dielectric layer between ferrite and metal leads to a strong enhancement in the propagation constant. It is also found that an axially magnetized homogeneous ferrite cylinder cannot support magnetostatic surface waves propagating along its curved surface in the azimuthal direction.**

I. Introduction

MAGNETOSTATIC wave propagation along curved ferrite surfaces is an area of importance on account of its relevance to a variety of practical situations, e.g., magnetostatic surface wave resonant modes of a ferrite slab with rounded edges [1], magnetic surface wave ring interferometer [2] characterized by propagation along the curved surface of a dielectric cylinder with ferrite sleeve, projected magnetostatic waveguide bends [3], scattering of electromagnetic waves by composite ferrite cylinders, etc. While the effect of curvature on guided wave propagation in hollow metallic [4], [5] and dielectric [6], [7] structures has been investigated in the past, similar studies in the case of ferrites are not available. In this paper, we have investigated the effect of curvature on propagation characteristics of magnetostatic surface waves in ferrites magnetized transverse to the direction of propagation. In Section II, the electromagnetic equations have been appropriately transformed and solved for the curved geometry. In Section III, the dispersion relation has been obtained in the general case of magnetostatic wave propagation, in azimuthal direction, and in a cylindrically curved

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The author is with the Department of Physics, Indian Institute of Technology, New Delhi, 110029, India.

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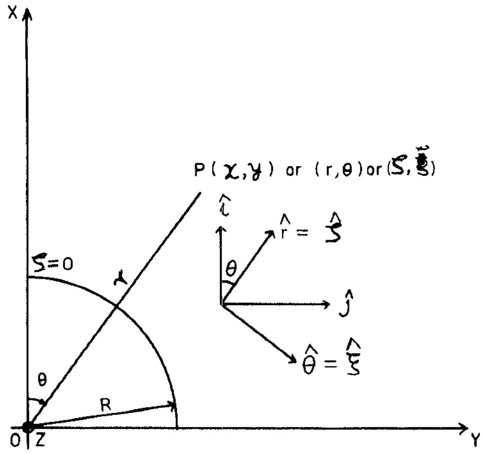

Fig. 1. The coordinate system.

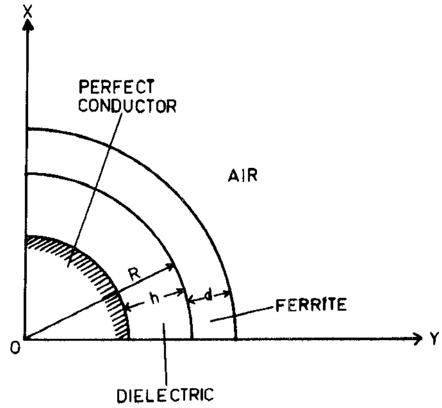

Fig. 2. The geometry of problem: curved ferrite slab backed by grounded dielectric.

slab of ferrite backed by a grounded layer of dielectric. The well known results [8]–[10] for the flat geometry follow as a special case when the radius of curvature approaches infinity. The cases of unbacked- and metal-backed curved ferrite slabs and of homogeneous ferrite cylinder have been specifically discussed. In the general case, the results of numerical calculations have been presented in Section III-C. It is found that the presence of a dielectric layer between the metallic cylinder and the ferrite sleeve leads to resonant enhancement in the propagation constant.

II. Magnetostatic Solution

The magnetic field h of the magnetostatic modes is described by the following equations:

$$\nabla\times h=0;\qquad h=\nabla\psi$$
$$\nabla.(\mu.h)=0 \tag{1}$$

where ψ is the magnetostatic potential and μ represents the permeability tensor of the ferrite magnetized along the z axis. In Cartesian coordinates, ψ can be shown [11] to satisfy the equation

$$\left[\mu(\partial^2/\partial x'+\partial^2/\partial y^2)+\partial^2/\partial z^2\right]\psi=0 \tag{2}$$

where μ is the leading diagonal element of the permeability tensor. Equation (2) may be expressed in a cylindrical coordinate system (Fig. 1) as

$$\left[\mu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right\}+\frac{\partial^2}{\partial z^2}\right]\subseteq(r,\theta,z)=0 \tag{3}$$

where $x=r\cos\theta$ and $y=r\sin\theta$. It can be shown that μ is invariant under this transformation; the components of b and h are related as

$$\begin{bmatrix}b_r\\b_\theta\\b_z\end{bmatrix}=\begin{bmatrix}\mu & ik & 0\\-ik & \mu & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}h_r\\h_\theta\\h_z\end{bmatrix} \tag{4}$$

where

$$h_r=\frac{\partial\psi}{\partial r},\qquad h_\theta=\frac{1}{r}\frac{\partial\psi}{\partial\theta},\qquad h_z=\frac{\partial\psi}{\partial z} \tag{5}$$

and, for a lossless ferrite,

$$\mu=\frac{H_0(H_0+4\pi M_0)-(\omega/\gamma)^2}{H_0^2-(\omega/\gamma)^2}$$
$$\kappa=\frac{1\pi M_0(\omega/\gamma)}{H_0^2-(\omega/\gamma)^2}. \tag{6}$$

In (6), H_0, ω, γ, and $4\pi M_0$ represent the biasing field, wave angular frequency, gyromagnetic ratio, and saturation magnetization, respectively. A transformation is made to variables (ζ,ξ) defined by

$$\zeta=R\ln(r/R),\qquad \xi=R\theta \tag{7}$$

whence

$$\frac{\partial}{\partial r}=\exp(-\zeta/R)\frac{\partial}{\partial\zeta},\qquad \frac{1}{r}\frac{\partial}{\partial\theta}=\exp(-\zeta/R)\frac{\partial}{\partial\xi}. \tag{8}$$

Substitution from (7) and (8) in (3) leads to

$$\left[\frac{\partial^2}{\partial\zeta^2}+\frac{\partial^2}{\partial\xi^2}+\exp(2\zeta/R)\frac{\partial^2}{\partial z^2}\right]\psi(\zeta,\xi,z)=0. \tag{9}$$

In most practical situations, the RF field is uniform along the dc magnetization (i.e., $\partial/\partial z=0$), in which case the magnetostatic potential for propagation along the curved surface is given by

$$\psi=\left[A\exp(\beta\zeta)+B\exp(-\beta\zeta)\right]\exp(-i\beta\xi) \tag{10}$$

where A and B are arbitrary constants. In problems involving layered structures, one has to match the magnetostatic potential and the normal component of magnetic induction at each interface. The latter is given by

$$b_\zeta=b_r=\beta\exp(-\zeta/R)\cdot\left[(\mu+\kappa)A\exp(\beta\zeta)-(\mu-\kappa)B\exp(-\beta\zeta)\right]\exp(-i\beta\xi). \tag{11}$$

III. Magnetostatic Surface Waves in a Curved Slab

A. Dispersion Relation

Consider the structure shown in Fig. 2 where R is the inner radius of the curvature of the curved ferrite slab, while d and h represent the thicknesses of the ferrite and dielectric, respectively. The metal-dielectric and dielectric-ferrite interfaces are defined by

$$\zeta = R\ln(1-h/R) \equiv \zeta_m \qquad (12)$$

and

$$\zeta = R\ln(1+d/R) \equiv \zeta_d \qquad (13)$$

respectively. On ignoring the variation of the wave field along the dc magnetization, the magnetostatic potential fields in the dielectric, ferrite, and air regions are given by

$$\psi^d = [A \exp(\beta\zeta) + B \exp(-\beta\zeta)] \exp(-i\beta\xi) \qquad (14)$$

$$\psi^f = [C \exp(\beta\zeta) + D \exp(-\beta\zeta)] \exp(-i\beta\xi) \qquad (15)$$

$$\psi^a = F \exp\{\beta(\zeta_d - \zeta)\} \exp(-i\beta\xi). \qquad (16)$$

Here A, B, C, D, and F are arbitrary constants. From (11), the normal components of magnetic induction may be written as

$$b_\zeta^d = \beta \exp(-\zeta/R)[A \exp(\beta\zeta) - B \exp(-\beta\zeta)] \cdot \exp(-i\beta\xi) \qquad (17)$$

$$b_\zeta^f = \beta \exp(-\zeta/R)[(\mu+\kappa)C \exp(\beta\zeta) - (\mu-\kappa)D \cdot \exp(-\beta\zeta)] \exp(-i\beta\xi) \qquad (18)$$

$$b_\zeta^a = -\beta F \exp(-\zeta/R) \exp\{\beta(\zeta_d - \zeta)\} \cdot \exp(-i\beta\xi). \qquad (19)$$

At the metal-dielectric interface ($\zeta = \zeta_m$), the normal component of magnetic induction should vanish and consequently

$$A \exp(\beta\zeta_m) = B \exp(-\beta\zeta_m). \qquad (20)$$

At the dielectric-ferrite interface ($\zeta = 0$), the matching of ψ's and b_ζ's results in

$$C + D = A + B \qquad (21)$$

$$(\mu+\kappa)C - (\mu-\kappa)D = A - B. \qquad (22)$$

At the ferrite-air interface ($\zeta = \zeta_d$), field matching leads to

$$C \exp(\beta\zeta_d) + D \exp(-\beta\zeta_d) = F \qquad (23)$$

$$(\mu+\kappa)C \exp(\beta\zeta_d) - (\mu-\kappa)D \exp(-\beta\zeta_d) = -F. \qquad (24)$$

All arbitrary constants may be eliminated from (20)–(24) to obtain the following dispersion relation for propagation in forward ($+\xi$) as well as reverse ($-\xi$) directions.

$$\exp(2\beta_\pm\zeta_d) = \frac{(\mu \mp \kappa - 1)[(\mu \pm \kappa + 1)\exp(2\beta_\pm\zeta_m) + (\mu \pm \kappa - 1)]}{(\mu \pm \kappa + 1)[(\mu \mp \kappa - 1)\exp(2\beta_\pm\zeta_m) + (\mu \mp \kappa + 1)]}. \qquad (25)$$

B. Special Cases

1) Flat Surface: In this case, $R \to \infty$ and $\zeta_d = R\ln(1 + d/R) \to d$, whereas $\zeta_m = R\ln(1-h/R) \to -h$ and (25) reduces, as expected, to the well known result for the dielectric layered planar structure, as derived by Bongianni [10], except for the notations.

2) Homogeneous Ferrite Cylinder: In this case, $h = 0$ and $d = -R$; $\exp(2\beta\zeta_m) \to 1$ and $\exp(2\beta\zeta_d) \to 0$. Consequently, no magnetostatic wave modes are supported by an axially magnetized, homogeneous ferrite cylinder, along its curved surface, in the azimuthal direction.

In experiments on magnetostatic surface waves in ferrite slabs with rounded edges, multiple round trip echoes are also observed [1] along with the first delayed MSSW signal. It is natural to conjecture [1] that a surface wave of sufficiently small wavelength propagates along the curvature to move to the other face of the slab and thereafter propagate in the opposite direction. However, according to the present analysis, such a wave cannot propagate around the curvature as a mode and, therefore, should be attenuated. This is quite understandable physically (at least in the case of small wavelengths) because a wave of small wavelength does not "see" the curvature of the ferrite and the latter behaves more or less like a flat half space; it is easy to show that a transversely magnetized half space does not support magnetostatic surface modes. Multiple echoes are observed in the experiment, perhaps because the wave, though attenuated, is not completely absorbed in one trip, while propagating around the curvature. A reduction in attenuation around the curvature should lead to the observation of a larger number of echoes. This may be achieved by inserting metallic or dielectric latches near the rounded edges; a ferrite cylinder with a metallic or dielectric latch can indeed support magnetostatic surface modes, as shown in what follows.

3) Dielectric Cylinder with Ferrite Sleeve: In this case, $h = R$ and hence $\zeta_m \to -\infty$; the dispersion relation (25) reduces to the following (reciprocal) form:

$$\exp(2\beta_\pm\zeta_d) = \frac{(\mu-\kappa-1)(\mu+\kappa-1)}{(\mu-\kappa+1)(\mu+\kappa+1)} \equiv f(\omega). \qquad (26)$$

This is the same as that for a flat unbacked-ferrite slab [8], [9], except for the presence of ζ_d instead of d. If β_0 denotes the propagation constant in the absence of curvature (other parameters unchanged), we have

$$\exp(2\beta\zeta_d) = f(\omega) = \exp(2\beta_0 d)$$

or

$$\beta/\beta_0 = \frac{d/R}{\ln(1+d/R)} > 1. \qquad (27)$$

Similarly, it can be shown that

$$\frac{(d\omega/d\beta)}{(d\omega/d\beta_0)} = \frac{\ln(1+d/R)}{d/R} < 1. \qquad (28)$$

Clearly, the effect of curvature is to reduce the group velocity as well as the phase velocity of MSSW's. However, when d is small ($\leqslant 0.01$ cm.), such as in the case of the magnetic surface wave ring interferometer [2], we obtain

$$(\beta-\beta_0)/\beta_0 \cong d/2R \tag{29}$$

which is quite small for typical values of d and R. Thus the effect of curvature would be insignificant unless $d/R \gtrsim 0.1$.

4) Metallic Cylinder with Ferrite Sleeve: In this case, $h=0$; $\zeta_m=0$ and the dispersion relation (25) is transformed into

$$\exp(2\beta_\pm \zeta_d) = \frac{(\mu \mp K-1)(\mu \pm K)}{(\mu \pm K+1)(\mu \mp K)} \equiv g_\pm(\omega) \tag{30}$$

which differs from the result derived earlier by Seshadri [9] in the case of a flat grounded ferrite slab only in that ζ_d appears instead of d. The consequences are similar to that for the case of the dielectric cylinder with a ferrite sleeve as discussed above.

C. General Case

The expressions for μ and K may be substituted from (6) into (25) to obtain

$$\exp(2\beta_\pm \zeta_d) = \frac{(2\pi M_0)^2 + 2\pi M_0(H_0+2\pi M_0 \mp \omega/\gamma)\exp(2\beta_\pm \zeta_m)}{\left[(H_0+2\pi M_0)^2-(\omega/\gamma)^2\right] + 2\pi M_0(H_0+2\pi M_0 \mp \omega/\gamma)\exp(2\beta_\pm \zeta_m)}. \tag{31}$$

We will discuss only the case when $d \ll R$, which is of interest to the ring interferometer [2]. When h is very small or very large, we obtain, in the limit, the cases of dielectric-ferrite and metal-ferrite cylinders discussed above; the effect of curvature on propagation constant is negligible. However, for intermediate values of h, β may be significantly different from β_0 even when d is small. To investigate this possibility, we have made numerical investigations of (31) for a typical set of parameters for YIG. It is found that the mode β_+ is practically unaffected by variations in h and, hence, shows only a weak dependence on R, given approximately by (29). However, the mode β_- is strongly influenced by the presence of the dielectric layer. Fig. 3 shows the fractional change in the propagation constant β, due to curvature, for various thicknesses (h) of the dielectric layer. For a given h, there exists a frequency range in which the fractional change in β is significantly large. In particular, the peak value may be orders of magnitude higher than the corresponding fractional change in the cases of dielectric-ferrite and metal-ferrite cylinders discussed in Section III-B. When h is relatively large, the peaks occur in the low-frequency region and the frequency bandwidth around the peak is small. When h is reduced, the region of interest broadens out and shifts towards higher frequencies. Moreover, the peak height is reduced. There is a critical value of h (~0.07 cm., for the present case) for which the peak fractional change in β is minimum (although still about 40 times the value for the cases of dielectric-ferrite and metal-ferrite cylinders) whereas the bandwidth of interest, around the peak, is maximum. When h is reduced beyond the critical value, once again the peak height increases and the region of interest narrows down, although it continues to shift towards higher frequencies. It is difficult to physically interpret the nature of these curves. In the limit of zero dielectric thickness, the fractional change in β reaches maximum when $\omega=\gamma(H_0+2\pi M_0)$, i.e., under the condition of magnetic resonance; this result seems plausible. However, for finite thickness of the dielectric, it appears that redistribution of energy due to curvature is responsible for the nature of these curves. A rigorous electromagnetic analysis is required for a better understanding.

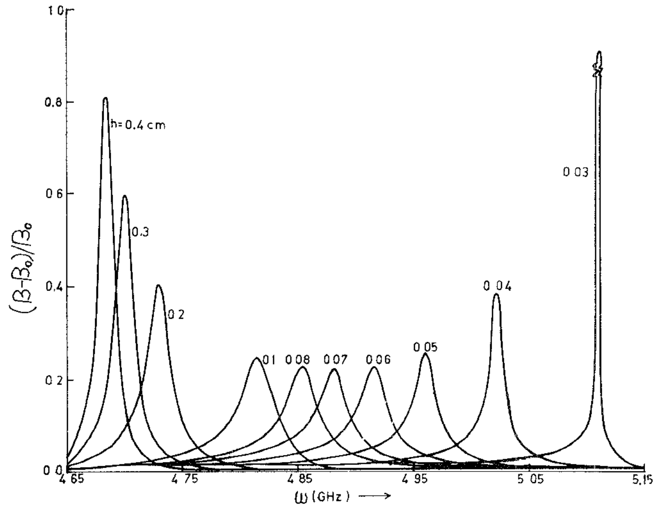

Fig. 3. Variation of the fractional change in the propagation constant $((\beta-\beta_0)/\beta_0)$ with frequency ω (expressed in GHz) for various thicknesses of the dielectric layer. Other parameters are: $H_0=1.0$ kG, $4\pi M_0=1.75$ kG, $\gamma=2.8$ GHz/kG, $d=0.01$ cm, and $R=1.0$ cm.

It is interesting to note that magnetostatic propagation along curved surface does not lead to losses other than absorption by the medium. This seems to violate the general principle that curvature of a guiding structure with a non-metallic boundary always leads to a radiation loss [6]. This apparent anomaly is due to the fact that magnetostatic approximation has been employed to obtain the modes; a rigorous electromagnetic analysis to the problem would certainly reveal losses due to curvature. However, in the region away from the cutoff, where magnetostatic approximation is approximately valid, the curvature losses are expected to be negligibly small. This is important from the view point of futuristic applications in magnetostatic waveguide bends [3].

IV. Summary

Magnetostatic equations have been rewritten in a suitable coordinate system and solved in the case of propagation along curved ferrite surfaces. It is found that an axially magnetized, homogeneous ferrite cylinder does not support magnetostatic surface waves propagating in the azimuthal direction, along the curved surface. The magnetostatic modes of a thin, curved ferrite slab (unbacked or metal backed) are not significantly affected when the slab thickness is a very small fraction of the radius of curvature. However, the presence of a grounded layer of dielectric on the inner side of the slab leads to a resonant enhancement in the fractional change in the propagation constant. In the region where magnetostatic approximation is approximately valid, the "curvature loss" is expected to be negligible.

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On the Resonant Frequency of a Reentrant Cylindrical Cavity

MAREK JAWORSKI

Abstract—**A new efficient method determining the resonant frequency of a reentrant cylindrical cavity is suggested. The method is based on solving the Helmholtz equation within two cavity regions and matching the solutions across the boundary surface. Contrary to similar formulations published previously, the continuity conditions on the boundary are imposed in a rigorous way. As a result, the solution is obtained in a form of successive approximations converging to the exact resonant frequency when a number of iterations tend toward infinity. Numerical examples are given for a few reentrant cavities of typical dimensions. Comparison is also made with experimental data as well as other theoretical results.**

I. INTRODUCTION

REENTRANT cylindrical cavities, widely used for a number of years, have recently found a new application in solid-state devices, particularly Gunn and tunnel diode oscillators. Simultaneously, a renewed interest in approximate methods determining the resonant frequency of such cavities has been observed. In some applications it is sufficient to consider a simple equivalent circuit, usually based on TEM coaxial line and lumped capacitance [1]–[5]. In general, however, more sophisticated methods are needed in order to evaluate the resonant frequency with reasonable accuracy [6]–[9].

Recently, a new interesting approach has been suggested by Williamson [9]. In his method, the magnetic field in both regions of the reentrant cavity is excited by the "aperture" electric field given on the interface $r=a$ (see Fig. 1). The resonant frequency is then found by matching the magnetic fields across the interface and solving the appropriate transcendental equation.

The above formulation, being in fact an improvement of Hansen's approach [6], is numerically simple and provides more accurate results than the solutions published previously. Nevertheless, the main disadvantage of both Williamson's and Hansen's method is due to the fact that the aperture field, which is generally not known, has to be included in the transcendental equation. In the paper of Williamson [9], the solution of the corresponding cylindrical antenna problem has been suggested as a suitable approximation for the electric field on the interface $r=a$. Unfortunately, such an approximation is sufficiently accurate for narrow-gap cavities only. Moreover, the solution of the antenna problem, as formulated for an unbounded region, may not be adequate for resonant systems, particularly in the cases when the outer diameter of

Manuscript received April 27, 1977; revised August 31, 1977.
The author is with the Institute of Physics, Polish Academy of Sciences, 02-668 Warsaw, Poland.

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